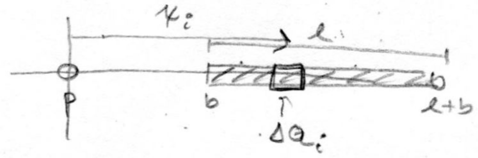
Continuous charge distributions and their electric field

Important constants: $\frac{1}{4\pi\epsilon_0} = 9\times10^9 \frac{N\cdot m^2}{C^2}$ $\quad \epsilon_0 = 8.85\times10^{-12} \frac{C^2}{Nm^2}$

1) Dimension $\left[\lambda(x) = \frac{\Delta Q}{\Delta x}\ \left[\frac{C}{m}\right]\right]$ $\left[Q_{tot} = \lambda L\right]$

consider a rod:



the contribution due to this element $\Delta Q_i = \lambda \Delta x_i \Rightarrow \Delta E_i = \frac{1}{4\pi\epsilon_0}\frac{\Delta Q_i}{x_i^2}$ ← but, whatever is x_i?

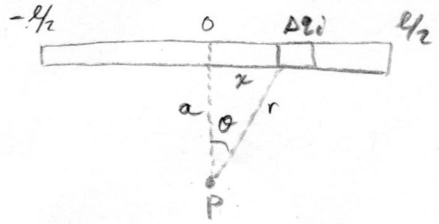
Sum up pieces / Integrate: $\sum \Delta E_i = \frac{\lambda}{4\pi\epsilon_0}\sum \frac{\Delta x_i}{x_i^2}$

$$\Downarrow$$

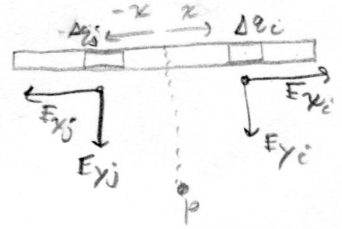
$$\boxed{E_p = \frac{\lambda}{4\pi\epsilon_0}\int_b^{\ell+b}\frac{dx}{x^2} = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{1}{b}-\frac{1}{\ell+b}\right) = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{\ell}{b(\ell+b)}\right)}$$ ← Q

as $b \gg L \Rightarrow \boxed{E_p \to \frac{Q}{4\pi\epsilon_0}\frac{1}{b^2}}$ <u>WE RECOVER $\vec{E}$ for point charge!</u>

what if p is not on axis of rod?

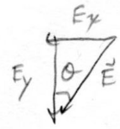


consider:



Just add vertical components of $\vec{E}$

$\Delta E_{iy} = \frac{1}{4\pi\epsilon_0}\frac{\Delta q_i}{r_i^2}(\cos\theta)$



! Horizontal components CANCEL (symmetry) $(\sum E_x = 0)$

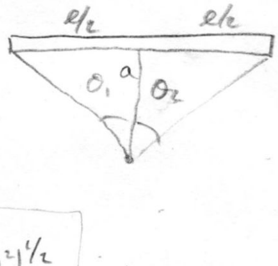
thus, $E_{p_y} = \frac{\lambda}{4\pi\epsilon_0}\int\frac{dx}{r^2}\cos\theta$ NO $\quad$ but $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$

and $\cos\theta = \frac{a}{r} \Rightarrow \frac{1}{r} = \frac{\cos\theta}{a}$

$$= \frac{\lambda}{4\pi\epsilon_0}\int\left(\frac{\cos^2\theta}{a^2}\right)(a\sec^2\theta\, d\theta)\cos\theta = \frac{\lambda}{4\pi\epsilon_0}\int_{\theta_1}^{\theta_2}\frac{\cos\theta\, d\theta}{a}$$ yes

But, what is θ_1, θ_2 or the angles corresponding to the ends of the rod?

Clearly $\theta_1 = -\theta_2$ with $\sin\theta = \frac{\ell/2}{\sqrt{a^2+\left(\frac{\ell}{2}\right)^2}}$



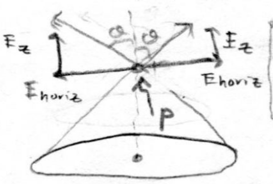
Performing the integration, we get

$E_{p_y} = \frac{\lambda}{4\pi\epsilon_0}\sin\theta_1 - \sin\theta_2 = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{2\sin\theta_1}{a}\right)$

$$\boxed{E_{p_y} = \frac{\lambda}{2\pi\epsilon_0 a}\left(\frac{\ell/2}{\sqrt{a^2+(\ell/2)^2}}\right) = \frac{Q}{2\pi\epsilon_0 a}\frac{1}{\left(a^2+\left(\frac{\ell}{2}\right)^2\right)^{1/2}}}$$

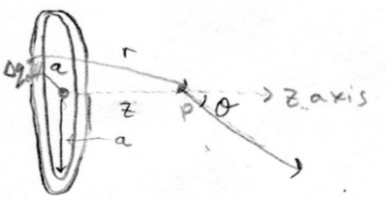
2-Dimensions

Suppose the rod is bent into a circle, and we want to find the electric field above its center.



[Like the 1D rod, we will get pairwise cancellations of the horizontal components. $\left(\sum E_{horiz} = 0\right)$]

So, we <u>ONLY NEED TO ADD THE VERTICAL $(\hat{k})$ components!</u>



$\Delta E_z = \frac{1}{4\pi\epsilon_0}\frac{\Delta q}{r^2}\cos\theta \Rightarrow E_z = \int \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\cos\theta$ but $dq = \lambda\, dl \ldots$

recall $\cos\theta = \frac{z}{r}$ $\quad r = \sqrt{z^2+a^2}$ $\quad$ where $a = \frac{l}{2\pi}$

$$\boxed{E_z = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda z\, dl}{(z^2+a^2)^{3/2}} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{(\lambda l)\, z}{(z^2+a^2)^{3/2}}}$$

if $z >> a$ $\quad E_z \rightarrow \frac{1}{4\pi\epsilon_0}\frac{\lambda l z}{(z^2)^{3/2}} = \frac{Q}{4\pi\epsilon_0}\frac{1}{z^2}$ ✓

Notes on discussion session

recall $\boxed{E_{P_y} = \frac{Q}{2\pi\epsilon_0}\frac{1}{a(a^2+(l/2)^2)^{1/2}}}$ if p is at the <u>midpoint</u> of a line charge distribution.

! <u>check units</u> $\Rightarrow$ $\frac{Q\ [C]}{2\pi\epsilon_0 \left[\frac{C^2}{N\cdot m^2}\right]}\frac{1}{a(a^2+(l/2)^2)^{1/2}}\left[\frac{1}{m^2}\right]$

After some cancellations, we get $\left[\frac{N}{C}\right]$ ✓